PERIODIC COHOMOLOGY: AN INTRODUCTION TO GROUP COHOMOLOGY

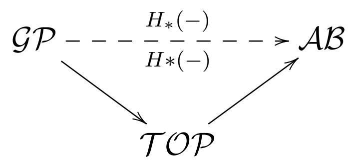
MIGUEL A. MALDONADO

Abstract. In these notes we offer an overview of the theory of groups having periodic cohomology. With this purpose the first sections are devoted to basic group cohomology, emphasizing the topological aspects of the theory. We offer no new material and so there are no proofs of the statements. Instead we offer references where the interested reader can go deeper in the theory. Some familiarity with basic notions of algebraic topology and module theory are taken for granted

Contents

1. Introduction

In general terms group homology and group cohomology are functors $H_*(-), H^*(-)$ from the category of groups (and homomorphisms of groups) $\mathcal{GP}$ to the category of abelian groups (and homomorphisms of abelian groups) $\mathcal{AB}$.



These functors can be constructed in purely algebraic terms or by means of certain functors from the category $\mathcal{GP}$ to the category $\mathcal{TOP}$ of topological spaces (and continuous functions). In this first section we will recall these functors and in later sections we will define the algebraic approach from group (co)homology.

1.1. (Co)homology via aspherical spaces.

A path-connected X is called ***aspherical*** if all its higher homotopy is trivial, that is $\pi_n X = 0, \forall n \geq 2$.

Theorem 1.1 (Hurewicz, '36, [9]). *The homotopy type of an aspherical space is totally determined by its fundamental group* $\pi_1 X$.

In particular, for $G = \pi_1 X$, this theorem states that the homotopy invariants of X can be thought as invariants of G. In particular, homology, cohomology and the Euler characteristic of X depends only on G and thus one can speak of the homology groups of G. An aspherical space X is also called an ***Eilenberg-MacLane space*** of the type $K(G, 1)$ and we define the homology and cohomology of the group G as

$$H_*(G, R) = H_*(X, R), \quad H^*(G, R) = H^*(X, R),$$

where the right side in both equations is cellular or singular (co)homology with coefficients R. Note that since X is determined by G, this definition does not depend on the space X.

A question arises at this point is it always possible to consider any group G as the fundamental group of certain space X? The following theorem says that we can.

Theorem 1.2 (Eilenberg-MacLane, '53). *For every group G there exists an aspherical CW-complex X such that $G = \pi_1 X$.*

This result is Corollary 1.28 in Hatcher's book [6] and is obtained as an application of the Seifert-Van Kampen Theorem on cell complexes. As a consequence of this theorem, the definition of $H_*(G)$ above can always be performed for any group.

The theorem above can be considered as a way of assigning to any group G the space $K(G,1)$ and it can be proved that for a homomorphism $\phi : G \to H$ there exists a continuous map $h : K(G,1) \to K(H,1)$ such that the homomorphism induced on homotopy groups coincides with ϕ (see Section 2.5 in [2]). Thus if consider an isomorphism $\phi : G \xrightarrow{\cong} H$ then the map $h : K(G,1) \to K(H,1)$ is an homotopy equivalence. All this makes evident that the association above is functorial:

$$\mathcal{GP} \longrightarrow \mathcal{TOP}, \qquad G \longmapsto K(G,1)$$

From the definition of group (co)homology above one can easily check

$$H_0(G;\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(G;\mathbb{Z}) \cong G_{ab} = G/[G,G],$$

where G_{ab} is the abelianization of G, that is, the quotient by $[G,G]$, the commutator subgroup of G. The group $H_2(G) = H_2(G;\mathbb{Z})$ was identified by Hopf ([8]) as an obstruction for the Hurewicz map $h_n : \pi_n(X) \to H_n(X)$ to be an isomorphism in dimension 2 by showing an exact sequence

$$\pi_2(X) \xrightarrow{h_2} H_2(X) \longrightarrow H_2(G) \longrightarrow 0$$

The identification of $H_1(G)$ as a quotient by the commutator is one of the ***Hop'fs formulae***, the second is

$$H_2(G) = \frac{R \cap [F,F]}{[R,F]},$$

for a short exact sequence $1 \to R \to F \to G \to 1$, with F free; that is, from a presentation of G in terms of generators and relations. See Section 2.5 of [4].

Example 1.3. *Consider a connected finite graph Γ. Contracting its maximal tree this graph becomes a wedge of circles (see Example 1.22* *in* [6]*). Therefore $\pi(\Gamma)$ is a free group and Γ is a $K(\pi_1(\Gamma), 1)$*

$$H_q(\pi_1(\Gamma)) = H_q(\Gamma) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^r, & q = 1 \\ 0, & q \geq 2 \end{cases}$$

where r is the number of edges of Γ that do not belong to the maximal tree.

Example 1.4. *The first $K(\mathbb{Z}, 1)$ space one meets at a first course in (algebraic) topology is S^1. Thus, the homology of $\mathbb{Z}$ is given by*

$$H_q(\mathbb{Z}) = H_q(S^1) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}, & q = 1 \\ 0 & q \geq 2 \end{cases}$$

(This can also be done as a particular case of the first example)

Example 1.5. *From the functorial properties mentioned early, a $K(G \times H, 1)$ space can be constructed as the product $K(G, 1) \times K(H, 1)$. Thus the torus $T^2 = S^1 \times S^1$ is of the type $K(\mathbb{Z} \times \mathbb{Z}, 1)$ and the homology of the product $\mathbb{Z} \times \mathbb{Z}$ is given by*

$$H_q(\mathbb{Z} \times \mathbb{Z}) = H_q(T^2) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z} \times \mathbb{Z}, & q = 1 \\ \mathbb{Z}, & q = 2 \\ 0 & q > 2 \end{cases}$$

Example 1.6. *If we consider the space X given as the wedge of r circles then $G = \pi_1 X$ is a free group F_r on r generators and $\pi_n X = 0, \forall n \geq 2$. Then*

$$H_q(F_r) = H_q(X) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^r & q = 1 \\ 0 & q \geq 2 \end{cases}$$

Example 1.7. *Let P^∞ be the infinite-dimensional real projective space defined as the quotient of the infinite-dimensional sphere S^∞ by the* *antipodal* $\mathbb{Z}_2$*-action. Moreover, since* S^∞ *is contractible it follows that* $\mathrm{P}^\infty = K(\mathbb{Z}_2, 1)$ *(see* [6]*, Section 1.B). Thus one has*

$$H^*(\mathbb{Z}_2) = H^*(\mathrm{P}^\infty) = \mathbb{Z}[\alpha]$$

where α *has dimension* 1*.*

As mentioned on the last example, the property of being aspherical can be stated in terms of the contractility of a covering of the space. We will recall this property on Section 2.2 devoted to resolutions.

1.2. **Via classifying spaces.** Let G be a topological group. Recall that a principal G-bundle over B is a fiber bundle $E \to B$ where each fiber is homeomorphic to G. There is an obvious functor $\mathcal{P}_G(X)$ which assigns to every CW-complex X the set of principal G-bundles over X. This functor is representable meaning that there exists a CW-complex BG such that there is a bijection

$$[X, BG] \cong \mathcal{P}_G(X),$$

where the left side is the set of homotopy classes of maps $X \to BG$. This space can be constructed using the ***Milnor's construction*** ([13]) which provide a universal principal G-bundle $EG \to BG$, where EG is a contractible space where G acts freely and $BG = EG/G$. If the group G is discrete then the space BG is an Eilenberg-MacLane $K(G, 1)$ with universal cover EG (Theorem 2.4.11 in [3]). This construction is functorial in the sense that for a group homomorphism $\phi : G \to H$ there is a map $h : BG \to BH$ whose homomorphism at the level of fundamental groups gives ϕ. As happened in the last section the space BG is unique up to homotopy equivalence [1]. We just have considered the functor

$$\mathcal{GP} \longrightarrow \mathcal{TOP}, \qquad G \longmapsto BG$$

We finally consider the homology and cohomology functors for spaces to define

$$H_*(G, R) = H_*(BG, R), \quad H^*(G, R) = H^*(BG, R)$$

[1]The construction of the G-bundle $EG \to BG$ produces an infinite dimensional space and BG has an infinite number of cells in each positive dimension if G is infinite. For example $B\mathbb{Z}$ is much bigger than S^1, which is a more efficient model for computations.

Example 1.8. *For $G = 1$, the trivial group, the one-point space $\{pt\}$ is a model for BG. Thus*

$$H^*(1) = H^*(\{pt\}) = \begin{cases} \mathbb{Z}, & * = 0 \\ 0, & * > 0 \end{cases}$$

as one knew.

Example 1.9. *Let G be the cyclic group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, then $BG = \mathrm{P}^\infty$ and one has*

$$H^*(\mathbb{Z}_2) = H^*(\mathrm{P}^\infty) = \mathbb{Z}[\alpha], \quad |\alpha| = 1,$$

as it was mentioned in the last section.

Example 1.10. *Let M be a oriented, closed, connected surface of genus $g \geq 1$ and note M is a model for $B\pi_1(M)$. Thus we have*

$$H_q(\pi_1(M)) = H_q(M) = \begin{cases} \mathbb{Z} & q = 0, 2 \\ \mathbb{Z}^{2g} & q = 1 \\ 0 & q > 2 \end{cases}$$

Example 1.11. *For $G = \mathbb{Z}_p$, with p odd, the classifying space BG is the* ***infinite lens space*** *and*

$$H^*(\mathbb{Z}_p) = H^*(BG) = \Lambda(x) \otimes \mathbb{F}_p[y]$$

the product of an exterior algebra and a polynomial algebra with $|x| = 1, |y| = 2$.

2. Resolutions

In this section we introduce the basic objects needed to give an algebraic approach to group (co)homology, the free and projective resolutions of a module. This object has the purpose of replacing an object (a module) by a sequence of objects (free or projective modules) that are easy to understand.

2.1. **Group rings.** The group ring of a group is a way of associating a ring to a group. In general terms, this ring can be considered as many copies of $\mathbb{Z}$ as the cardinality of G. The ***(integral) group ring*** $\mathbb{Z}G$ of a group G is the free abelian group generated by the elements of G as basis; that is, it consists of formal expressions

$$\sum \lambda_i g_i,\ g_i \in G$$

with $\lambda_i = 0$ for almost all $\lambda_i \in \mathbb{Z}$. It has the structure of a ring since the multiplication on G induces the operations $\cdot, + : \mathbb{Z}G \times \mathbb{Z}G \longrightarrow \mathbb{Z}G$, given by

$$\sum_i \lambda_i g_i + \sum_i \rho_i g_i = \sum_i (\lambda_i + \rho_i) g_i,$$

$$(\sum_i \lambda_i g_i) \cdot (\sum_i \rho_i g_i) = \sum_i (\sum_{g_j g_k = g_i} \lambda_i \rho_i) g_i$$

Note $(\mathbb{Z}G, +)$ is an abelian group with $\sum 0 g_i$ as identity element. Moreover, it follows easily that $(\mathbb{Z}G, +, \cdot)$ is a ring. For every $g_i \in G$ we can write

$$g_i = \sum \lambda_i g_i$$

by puttting $\lambda_i = 0$ for $i \neq j$ and $\lambda_j = 1$, that is, G is contained in $\mathbb{Z}G$ as subgroup. Moreover, if G is not abelian then $\mathbb{Z}G$ is not commutative.

The group ring has the following universal property which characterizes it.

Theorem 2.1 (Universal Property of Group Rings)**.** *Let R be a ring with 1_R and $\varphi : G \to R$ a function such that $\varphi(1) = 1_R$, $\varphi(g_i g_j) = \varphi(g_i)\varphi(g_j)$. Then there exists a unique ring homomorphism $\phi : \mathbb{Z}G \to R$ such that the following diagram commutes*

$$\begin{array}{ccc} G & \longrightarrow & \mathbb{Z}G \\ & \underset{\varphi}{\searrow} & \downarrow \phi \\ & & R \end{array}$$

where the horizontal map is the natural inclusion.

Example 2.2. *For G the trivial group one has $\mathbb{Z}G \cong \mathbb{Z}$, that can be checked direct from the definition.*

Example 2.3. *For the cyclic group $G = \mathbb{Z}/n =< \tau >$ the powers $1, \tau, \dots \tau^{n-1}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}G$. Since $\tau^n = 1$ we have*

$$\mathbb{Z}G \cong \mathbb{Z}[\tau]/(\tau^n - 1)$$

Example 2.4. *If G is the infinite cyclic group with generator τ, the ring group $\mathbb{Z}G$ has as basis the set of* ***Laurent polynomials*** *$\sum_{i \in \mathbb{Z}} a_i \tau^i$ where $a_i \in \mathbb{Z}$ with $a_i = 0$ for almost i. Notation: $\mathbb{Z}G = \mathbb{Z}[\tau, \tau^{-1}]$.*

Consider the function $\varphi : G \to \mathbb{Z}$, $g \mapsto 1$. From Theorem 2.1 there is a unique ring homomorphism $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ called the ***augmentation homomorphism***; note $\epsilon(g_i) = 1$ and $\epsilon(\sum \lambda_i g_i) = \sum \lambda_i$.

A ***G-module*** M is simply a $\mathbb{Z}G$-module M, in other terms is an abelian group M together with an action of G on M. Every abelian group M can have a trivial structure as G-module be defining $ga = a$, for $g \in G$ and $a \in M$; in this situation we say that M is a ***trivial*** G-module. In the theory that follows the case of interest will the integers $\mathbb{Z}$ having this trivial structure as module.

2.2. **Resolutions.** Let R be a ring with identity. A ***resolution*** $\mathcal{P}$ of a R-module M is an exact sequence of R-modules

$$\mathcal{P} : \qquad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

For a projective resolution $\mathcal{P}$ of M we call the sequence obtained from $\mathcal{P}$ by deleting M

$$\mathcal{P}_M : \qquad \cdots \longrightarrow P_2 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \longrightarrow 0$$

a ***reduced projective resolution*** of M. Note that with $\mathcal{P}_M$ we do not loose information since $M \cong coker\delta_1$ and the reason for considering $\mathcal{P}_M$ is that it consists exclusively of projective modules.

If the modules F_i are free the resolution is called a ***free resolution*** and if the modules are projective [2] the resolution is called a ***projective resolution***.

Proposition 2.5. *Every R-module has a free resolution.*

Proof. Recall that every R-module M is the quotient of a free R-module (Theorem 2.35 in [14]) thus there is an exact sequence

$$0 \to M_0 \xrightarrow{\alpha_0} F_0 \xrightarrow{\beta_0} M \to 0$$

where F_0 is free (this sequence is called a ***free presentation*** of M). Since M_0 is a quotient of a free R-module, there is an exact sequence

$$0 \to M_1 \xrightarrow{\alpha_1} F_1 \xrightarrow{\beta_1} M_0 \to 0$$

with F_1 free. In general, we obtain an exact sequence

$$0 \to M_n \xrightarrow{\alpha_n} F_n \xrightarrow{\beta_n} M_{n-1} \to 0$$

with F_n free. We define a sequence

$$\cdots \longrightarrow L_{n+1} \xrightarrow{\delta_{n+1}} L_n \xrightarrow{\delta_n} L_{n-1} \longrightarrow \cdots$$

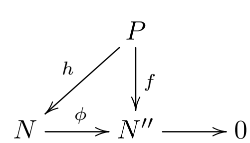
where

$$L_n = \begin{cases} M, & n = -1 \\ L_n, & n \geq 0 \\ 0, & n < -1 \end{cases}, \quad \delta_n = \begin{cases} \beta_0, & n = 0 \\ \alpha_{n-1} \circ \beta_n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Finally, since α is injective and β epimorphism, it follows that

$$im\delta_{n+1} = im\alpha_n = \ker \beta_n = \ker \delta_n$$

[2]A module P is called ***projective*** if for every homomorphism $f : P \to N''$ and every epimorphism $\phi : N \to N''$ there exists $h : P \to N$ such that $\phi \circ h = f$:



and the sequence is exact. □

The resolution constructed above is unique up to chain homotopy equivalence. The proof of this fact involves the construction of the chain maps and homotopies step by step (see Section I.7 in [4]). Since every free module is projective one gets that ***every module has a projective resolution***.

The main interest on (projective and free) resolutions is in the case of $R = \mathbb{Z}G$ and $M = \mathbb{Z}$ which reflects a quite natural topological phenomenon as the following example shows.

Example 2.6. *Let X be a CW-complex. The augmented cellular chain complex $C_*(X)$ of X is given by*

$$\cdots \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

*where $C_k(X) = H_k(X^k, X^{k-1})$ is the free abelian group with one generator for each k-cell and ϵ is the augmentation homomorphism. If X is a **G-complex** (that is G acts on X by permuting the cells) then G acts on $C_k(X)$, for every k. Thus $C_*(X)$ is a chain complex of G-modules.*

The CW-complex X is called a ***free G-complex*** if the action of G is free; that is, G freely permutes the cells of X: $g\sigma \neq \sigma, \forall \sigma, g \neq 1$. In this case every $C_k(X)$ has a $\mathbb{Z}$-basis freely permuted by G and thus $C_k(X)$ is a free $\mathbb{Z}G$-module with one basis element for every G-orbit of cells. Finally recall that if X is a contractible CW-complex, then X has the homology of a point and thus the sequence

$$\cdots \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

is exact. Thus, we have proved the following

Proposition 2.7. *If X is a contractible free G-complex, the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over the ring $\mathbb{Z}G$.*

Free G-complexes arises naturally in the theory of regular coverings which we recall in the following subsection

Detour on regular coverings. Let $p : \tilde{Y} \to Y$ be a covering of connected and locally path-connected spaces. Recall that a deck transformation of the covering p is a homeomorphism $g : \tilde{Y} \to \tilde{Y}$ such that $p \circ g = p$. The set of deck transformations forms a group G which acts freely on $\tilde{Y}$.

The covering p is called ***regular*** if the image of the induced homomorphism $p_* : \pi_1 \tilde{Y} \to \pi_1 Y$ is normal in $\pi_1 Y$. Equivalently, the covering is regular if G acts transitively on the fibers $p^{-1}(x)$, and thus $Y = \tilde{Y}/G$. For regular coverings one has $G \cong \pi_1 Y / \pi_1 \tilde{Y}$ and if p is the univesal covering (meaning $\tilde{Y}$ is simply connected) then one has $G \cong \pi_1 Y$.

Now, if the covering p is regular and Y has a structure of CW-complex, then $\tilde{Y}$ inherits a CW-structure where the group G of deck transformations acts by permutation of cells, here the open cells of $\tilde{Y}$ over an open cell σ of Y are the connected components of $p^{-1}(\sigma)$. Thus $\tilde{Y}$ is a free G-complex and $C_*(\tilde{Y})$ is a complex of free $\mathbb{Z}G$-modules with one basis element for each cell of Y. With this in hand there is another way to rephrase Proposition 2.7 above

Proposition 2.8. *For a $K(G,1)$ space, the augmented cellular chain complex of its universal cover is a free resolution of $\mathbb{Z}$ over the ring $\mathbb{Z}G$.*

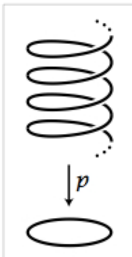
2.3. **Two more examples.** Consider the $G = F_S$ the free group generated by a set S and recall this is the fundamental group of the space Y given by a wedge sum of copies of circles S^1_s indexed by $s \in S$. The space Y is a CW-complex of dimension 1 with one 0-cell (the vertex) and one 1-cell for each element in S. By a dimension argument, note that Y is a $K(F_S, 1)$ space.

As a basepoint of the universal cover $\tilde{Y}$ of Y let us consider a vertex x_0 which represents the unique G-orbit of vertices of $\tilde{Y}$; thus it generates the free $\mathbb{Z}G$-module $C_0(\tilde{Y})$. For a basis of $C_1(\tilde{Y})$ we take, for each $s \in S$, an oriented 1-cell e_s of $\tilde{Y}$ over the copy S^1_s. Since the G-action is free, one can consider the initial vertex of e_s is the basepoint x_0 and the final vertex of e_s is sx_0 so that $\delta e_s = sx_0 - x_0 = (s-1)x_0$. In this case the free resolution of $\mathbb{Z}$ is given by

$$0 \longrightarrow \mathbb{Z}G_S \xrightarrow{\delta} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}G_S$ is a free $\mathbb{Z}G$-module with basis $\{e_s\}, \delta e_s = s - 1$ and $\epsilon(g) = 1$.

Consider the situation above with $S = \{\tau\}$. Thus G is the infinite cyclic group generated by τ with $\mathbb{Z}G = \mathbb{Z}[\tau, \tau^{-1}]$. The resolution above now has the form



$$0 \longrightarrow \mathbb{Z}[\tau, \tau^{-1}] \xrightarrow{\delta} \mathbb{Z}[\tau, \tau^{-1}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where δ is given by multiplication by $\tau - 1$ and ϵ can be described as an evaluation map: $f \mapsto f(1)$, for $f(\tau) \in \mathbb{Z}[\tau, \tau^{-1}]$. Note that in this case, the space Y is just a circle S^1 and thus its universal cover is the real line $\mathbb{R}$, visualized as an helix in $\mathbb{R}^3$ as shown in the figure.

2.4. **Periodic resolutions via actions on spheres.** Consider a G-complex X homeomorphic to an odd-dimensional sphere S^{2k-1}. Its cellular chain complex does not give a free resolution of $\mathbb{Z}$ as done above since X is not contractible, neverthless its chain complex can be used to construct one. Recall X has homology

$$H_q(X) = \begin{cases} \mathbb{Z}, & q = 0, 2k-1 \\ 0, & \text{otherwise} \end{cases}$$

The chain complex thus has the form

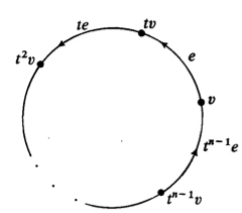
$$0 \longrightarrow \mathbb{Z} \longrightarrow C_{2k-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where every $C_i(X)$ is free and ϵ is the augmentation map. In order to get the desired free resolution we must "eliminate" the $\mathbb{Z}$ in position $2k$ by the following procedure:

$$\begin{array}{ccccccccccccc} \cdots \longrightarrow & C_1 & \longrightarrow & C_0 & \dashrightarrow & C_{2k-1} & \longrightarrow \cdots \longrightarrow & C_0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & & \downarrow & \nearrow & & & & & & & \\ & & & \mathbb{Z} & & & & & & & & \end{array}$$

Note this resolution is ***periodic*** of length $2k$ and continues forever to the left.

As a special case of the above consider the finite cyclic group of order n generated by τ $G = \mathbb{Z}/n\mathbb{Z} =< \tau >$ acting by rotations on S^1, (which can be) considered as a CW-complex with n vertices and n 1-cells as shown. The chain complex has the form



$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}G \xrightarrow{\tau-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where the homomorphism η is given by

$$\eta(1) = \sum_{i=0}^{n-1} \tau^i = 1 + \tau + \tau^2 + \cdots + \tau^{n-1} = N$$

and is called the ***norm element*** of $\mathbb{Z}G$. The periodic free resolution is

$$\cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{\tau-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\tau-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

Here the maps $N, \tau - 1$ denote multiplication by N and $\tau - 1$, respectively.

This is the first example of the phenomenon of periodicity arising from the free action of a group on a sphere, we will recall and generalize this result in later sections.

3. Derived functors

In order to have an algebraic definition for group (co)homology we will recall here the theory of derived functors from which the (co)homology of a group can be obtained by specializing on certain functors Tor and Ext , whose definition and main properties are in the next subsections.

Let R be a non necessarly commutative ring with unit. A (covariant or contravariant) functor $F : \mathcal{C} \to \mathcal{D}$ is called ***additive*** if it preserves the sum operation on morphism, that is, for every pair of morphisms f, g in $\mathcal{C}$, $F(f+g) = F(f) + F(g)$.

Example 3.1. *For R-modules N, M define the functors*

$$- \otimes N : M \longmapsto M \otimes N, \qquad M \otimes - : N \longmapsto M \otimes N$$

given by the tensor product of modules. Note this are additive covariant functors.

Example 3.2. *For R-modules N, M define the functors*

$$Hom(-, N) : M \longmapsto Hom(M, N), \qquad Hom(M, -) : N \longmapsto Hom(M, N)$$

given by the tensor product of modules. Note this are additive contravariant functors.

Let $F : \mathcal{MOD} \to \mathcal{AB}$ be an additive covariant functor from the category of modules on the category of abelian groups. The ***left derived functor*** of F (of degree n) is the functor defined in each module M as

$$L_n F(M) = H_n(F(\mathcal{P}_M)),$$

where $\mathcal{P}_M$ is a reduced projective resolution of M. That is, for every covariant functor F we define a series of functors $L_n F(-)$ by considering the resolution $\mathcal{P}_M$, applying F to this resolution and then considering the homology of the resulting sequence. Thus, a left derived functor measures the failure of the functor F of taking exact sequences to exact sequence, that is, it measures the ***exactness*** of functor F. Two properties of derived functors follows immediately from the definition:

- The functor $L_n F(M)$ does not depend on the reduced projective resolution of M.
- For every n, $L_n F(-) : \mathcal{MOD} \to \mathcal{AB}$ is an additive functor.

As above we define the ***right derived functor*** of a contravariant functor G (of degree n) as the functor defined by

$$R^n G(M) = H^n(G(\mathcal{P}_M)),$$

where $\mathcal{P}_M$ is a reduced projective resolution of M. This produces an additive functor and its definition does not depend on the choosen resolution.

In the next subsections we present the definition of two prominent examples of derived functors that serve to define the (co)homology of a group.

3.1. **The Tor functor.** Consider the additive covariant functor $-\otimes N$, M an R-module and $\mathcal{P}_M$ a reduced projective resolution of M. The left derived functor of this is called the ***torsion functor of*** M ***and*** N ***over*** R:

$$\boxed{\mathrm{Tor}_n^R(M,N) = L_n(-\otimes N)(M) = H_n(\mathcal{P}_M \otimes N)}$$

That is, from the resolution $\mathcal{P}_M$ we consider

$$\mathcal{P}_M \otimes N: \qquad \cdots \longrightarrow P_2 \otimes N \longrightarrow P_1 \otimes N \longrightarrow P_0 \otimes N \longrightarrow 0$$

and since this sequence is semi-exact we consider its homology modules.In the following we recall the main properties of the functor Tor.

Proposition 3.3. (1) *The functor Tor depends on* M, N *and* n *and does not depend on the chosen resolution.*
(2) $Tor_n^R(-,-)$ *is a covariant bifunctor on the category of* R*-modules on itself.*
(3) *For a short exact sequence of modules* $N' \hookrightarrow N \rightharpoonup N''$ *there is a long exact sequence*

$$\cdots \longrightarrow Tor_n^R(M,N') \longrightarrow Tor_n^R(M,N) \longrightarrow Tor_n^R(M,N'') \longrightarrow$$

$$\longrightarrow Tor_{n-1}^R(M,N') \longrightarrow \cdots \longrightarrow Tor_0^R(M,N'') \longrightarrow 0$$

(4) *For a short exact sequence of modules* $M' \hookrightarrow M \rightharpoonup M''$ *there is a long exact sequence*

$$\cdots \longrightarrow Tor_n^R(M',N) \longrightarrow Tor_n^R(M,N) \longrightarrow Tor_n^R(M'',N) \longrightarrow$$

$$\longrightarrow Tor_{n-1}^R(M',N) \longrightarrow \cdots \longrightarrow Tor_0^R(M'',N) \longrightarrow 0$$

(5) *For* $Tor_n^R(-,N)$ *we considered the functor* $-\otimes N$*. If we consider* $M\otimes -$ *for constructing* $Tor_n^R(M,-)$ *we obtain isomorphic groups.*
(6) $Tor_0^R(M,N)\cong M\otimes N$ *(natural equivalence between functors)*
(7) *For a projective module* P *one has*

$$Tor_n^R(M,P) = Tor_n^R(P,N) = 0, \quad \forall n \geq 1$$

Consider an exact sequence $0\to L\to M\to N\to 0$ and apply the functor $-\otimes D$ to obtain an exact sequence of abelian groups:

$$L\otimes_R D \longrightarrow M\otimes_R D \longrightarrow N\otimes_R D \longrightarrow 0. \tag{1}$$

By parts 3 and 6 of the Proposition 3.3 above we can extend this sequence to the left in the following way

$$\cdots \longrightarrow \mathrm{Tor}_2^R(N,D) \longrightarrow \mathrm{Tor}_1^R(L,D) \longrightarrow \mathrm{Tor}_1^R(M,D) \longrightarrow$$

$$\mathrm{Tor}_1^R(N,D) \longrightarrow L\otimes D \longrightarrow M\otimes D \longrightarrow N\otimes D \longrightarrow 0$$

and thus $\mathrm{Tor}_1^R(N,D)$ measures the failure of extending the sequence 1 to an exact sequence. The following result clarifies the name for Tor.

Proposition 3.4. *Let* A,B *be* $\mathbb{Z}$*-modules and consider* $t(A),t(B)$ *their torsion submodules. Then one has* $Tor_1^{\mathbb{Z}}(A,B) = Tor_1^{\mathbb{Z}}(t(A),t(B))$

3.2. **The Ext functor.** Consider the contravariant additive functor $\mathrm{Hom}(-,N)$, M an R-module and $\mathcal{P}_M$ a reduced projective resolution. The right derived functor is called the ***extension functor*** of M and N over R.

$$\boxed{\mathrm{Ext}_R^n(M,N) = R^n(\mathrm{Hom}(-,N))(M) = H^n(\mathrm{Hom}(\mathcal{P}_M,N))}$$

That is, from $\mathcal{P}_M$ we consider the sequence

$$0 \longrightarrow \mathrm{Hom}(P_0,N) \longrightarrow \mathrm{Hom}(P_1,N) \longrightarrow \cdots \longrightarrow \mathrm{Hom}(N,P_n) \longrightarrow \cdots$$

and take its cohomology.

As in the case of the torsion functor note that in the left side, there is no reference to the resolution $\mathcal{P}_M$ and the reason is that the definition above does not depend on it: if we change the resolution, we obtain isomorphic Ext groups. (In order to prove this one needs to compare resolutions and in this part one uses that projective modules can be used to *lift* homomorphisms. See Proposition 4, Section 17.1 of [5])

Proposition 3.5. (1) *$Ext^n_R(-,-)$ is a bifunctor from the category of R-modules into itself; it is contravariant on the first coordinate and covariant on the second*

(2) *For a short exact sequence of modules $N' \hookrightarrow N \rightharpoonup N''$ there is a long exact sequence*

$$0 \longrightarrow Ext^0_R(M,N') \longrightarrow Ext^0_R(M,N) \longrightarrow Ext^0_R(M,N'') \longrightarrow$$

$$\cdots \longrightarrow Ext^n_R(M,N') \longrightarrow Ext^n_R(M,N) \longrightarrow Ext^n_R(M,N'') \longrightarrow \cdots$$

(3) *For a short exact sequence of modules $M' \hookrightarrow M \rightharpoonup M''$ there is a long exact sequence*

$$0 \longrightarrow Ext^0_R(M'',N) \longrightarrow Ext^0_R(M,N) \longrightarrow Ext^0_R(M',N) \longrightarrow$$

$$\cdots \longrightarrow Ext^n_R(M'',N) \longrightarrow Ext^n_R(M,N) \longrightarrow Ext^n_R(M',N) \longrightarrow \cdots$$

(4) *$Ext^0_R(M,N) \cong Hom_R(M,N)$*

(5) *$Ext^n_R(P,N) = 0$, for every projective module P.*

Recall that a short exact sequence $0 \to L \to M \to N \to 0$ gives rise to an exact sequence of abelian groups

$$0 \longrightarrow \mathrm{Hom}_R(N,D) \longrightarrow \mathrm{Hom}_R(M,D) \longrightarrow \mathrm{Hom}_R(L,D). \tag{2}$$

In general the homomorphism on the right is not surjective and thus the sequence above cannot always be extended to a short exact sequence. By parts 3 and 4 of the Proposition 3.5 above one has an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(N,D) \longrightarrow \mathrm{Hom}_R(M,D) \longrightarrow \mathrm{Hom}_R(L,D) \longrightarrow$$

$$\longrightarrow \mathrm{Ext}^1_R(N,D) \longrightarrow \mathrm{Ext}^1_R(M,D) \longrightarrow \mathrm{Ext}^1_R(L,D) \longrightarrow \cdots$$

and thus $\mathrm{Ext}^1_R(N,D)$ measures the failure of the sequence 2 above to be extended to an exact sequence. In particular, if $\mathrm{Ext}^1_R(N,D) = 0$ for all R-modules N then the sequence 2 is always exact on the right. Finally, we consider a result that justifies the terminology for the functor Ext.

Proposition 3.6. *Let N, L be R-modules. Then there is a bijection between $Ext^1_R(N,L)$ and the set of equivalence classes of extension of N by L.*

This result will be considered in the next section when we define the cohomology of a group in terms on the Ext groups.

4. Group cohomology

Let $\mathcal{P}_{\mathbb{Z}}$ be a reduced projective resolution of $\mathbb{Z}$ over the group ring $\mathbb{Z}G$. For a G-module N we define the ***homology of*** G ***in dimension*** n ***with coefficients in*** N as [3]

$$\boxed{H_n(G;N) = \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, N)}$$

In the same fashion we define the ***cohomology of*** G ***in dimension*** n ***and coefficients in*** N as

$$\boxed{H^n(G;N) = \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, N)}$$

In both definitions $\mathbb{Z}$ is considered as a trivial G-module and the module N is called the ***coefficient module***. By the properties of Tor and Ext described in the last section we see that both H_n, H^n are both covariant functors on the coefficients module: for homomorphism $f : N \to N'$ there are induced homomorphisms

$$H_n(G;N) \longrightarrow H_n(G;N'), \quad H^n(G;N) \longrightarrow H^n(G;N')$$

Also from the definition it follows that H^n is a contravariant functor and H_n covariant: for a group homomorphism $\phi : G \to G'$ there are induced homomorphisms

$$H_n(G;N) \longrightarrow H_n(G';N), \quad H^n(G';N) \longrightarrow H^n(G;N)$$

for a G-module N.

Before we go further on the properties of group (co)homology we consider the definition of $H_n(G), H^n(G)$ from an Eilenberg-MacLane spaces and we will prove the following

Theorem 4.1. *There are canonical isomorphisms*

$$H_n(G,R) = H_n(K(G,1),R),$$
$$H^n(G,R) = H^n(K(G,1),R)$$

This is Theorem 2.2.3 in [3] and we will follow that proof based mainly on Proposition 2.8 on Section 3 but considering the bifunctors Ext and Tor.

Proof. Consider a CW-complex $X = K(G,1)$, $\tilde{X}$ its universal cover and the group G acting on X by permuting the cells. Thus Galso acts

[3]In fact this definition corresponds to a *left* module N but we assume all modules involved have both left and right actions, that is the case when one considers the group ring $\mathbb{Z}G$ via the automorphism $g \mapsto g^{-1}$ and setting $mg = g^{-1}m$.

on $\tilde{X}$ by permuting the cells and its associated cellular chain complex $C_*(\tilde{X};R)$ is an exact sequence of free G-modules since $\tilde{X}$ is contractible. We finally get a free resolution of R as an G-module by considering the augmentation map ϵ:

$$\cdots \to C_n(\tilde{X};R) \to \cdots C_1(\tilde{X};R) \to C_0(\tilde{X};R) \xrightarrow{\epsilon} R \to 0$$

Note $C_i(\tilde{X};R) \otimes_G R \cong C_i(X;R)$ and thus one has:

$$H_i(G,R) = \mathrm{Tor}_i^{\mathbb{Z}G}(R,R) \cong H_i(X,R).$$

In a similar way, considering the functor Hom one has that $\mathrm{Hom}_{\mathbb{Z}G}(C_i(\tilde{X},R),R) \cong \mathrm{Hom}_R(C_i(X;R),R) = C^i(X;R)$ and so

$$H^i(G,R) = \mathrm{Ext}^i_{\mathbb{Z}G}(R,R) \cong H^i(X,R). \quad \square$$

In the case of the definition given in terms of the classifying space BG we use the theorem above and Theorem 2.4.11 part i) in [3].

As it happens with topological spaces only basic calculations can be made direct from the definition of cohomology. Instead one better looks for general properties of the group (co)homology and used them in practice. In what follows we will recall these properties.

- For a short exact sequence of G-modules $N' \hookrightarrow N \rightharpoonup N''$ and every $n > 0$ there is a ***connecting homomorphism*** $\delta : H_n(G,N'') \to H_{n-1}(G,N')$ such that there is a long exact sequence in homology

$$\cdots \to H_n(G,N') \to H_n(G,N) \to H_n(G,N'') \xrightarrow{\delta} H_{n-1}(G,N') \to \cdots$$

$$\cdots \to H_1(G;N'') \xrightarrow{\delta} H_0(G,N') \to H_0(G,N) \to H_0(G,N'') \to 0$$

- For a short exact sequence of G-modules $N' \hookrightarrow N \rightharpoonup N''$ and every $n > 0$ there is a ***connecting homomorphism*** $\delta : H^n(G,N'') \to H^{n+1}(G,N')$ such that there is a long exact sequence in cohomology

$$0 \to H^0(G;N') \to H^0(G;N) \to H^0(G;N'') \xrightarrow{\delta} H^1(G,N') \to \cdots$$

$$\cdots \to H^n(G;N') \to H^n(G,N) \to H^n(G,N'') \xrightarrow{\delta} H^{n+1}(G,N') \to \cdots$$

- Define the ***invariant subgroup*** N^G of the G-module N as

$$N^G = \{n \in N \mid gn = n, \forall g \in G, n \in N\}$$

It is not hard to proof that N^G is the maximal submodule of N where G acts trivially. There is an identification $\boxed{H^0(G;N) = N^G}$ arising from the isomorphism $N^G \cong$

$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, N)$. Thus, if N is a trivial G-module, one has $H^0(G; N) = N$.

- Define the ***coinvariants*** N_G of the G-module N as

$$N_G = N/ < \{gn - n \mid g \in G, n \in N\} >$$

Here N_G is the largest quotient module of N having a trivial action of G and we have the isomorphism $N_G \cong \mathbb{Z} \otimes N$. Thus there is an identification $\boxed{H_0(G; N) = N_G}$. If the G-action on N is trivial, one has $H_0(G; N) = N$.

- The group $H_1(G; N)$ can be identified with $N \otimes G/[G, G]$ (Section VI.4 of [7]), where $[G, G]$ is the submodule generated by all the elements $x^{-1}y^{-1}xy$ (the ***commutator subgroup*** of G). From this one gets the isomorphism

$$\boxed{H_1(G; \mathbb{Z}) \cong G/[G, G] = G_{ab},}$$

where $\mathbb{Z}$ is considered as a trivial G-module.

- For cohomology one has

$$\boxed{H^1(G; N) \cong \mathrm{Hom}(H_1(G; \mathbb{Z}), N)}$$

- An ***extension of*** G ***by a module*** A is an exact sequence $A \hookrightarrow E \rightharpoonup G$. Two such extensions are called ***equivalent*** if there is an isomorphism ϕ such that the following diagram commutes

$$\begin{array}{ccccc} A & \longrightarrow & E & \longrightarrow & G \\ \| & & \downarrow{\scriptstyle \phi} & & \| \\ A & \longrightarrow & E' & \longrightarrow & G \end{array}$$

It turns out that if $\mathcal{E}(G, A)$ denotes the set of equivalence classes of extensions of G by A, then there is a 1-1 correspondence between $\mathcal{E}(G, A)$ and $H^2(G, A)$.

4.1. **The cup product.** Let G, G' be groups, M an G-module, M' an G'-module and $\mathcal{P}, \mathcal{P}'$ projective resolutions of $\mathbb{Z}$ over the group ring $\mathbb{Z}G$. It is possible to construct maps

$$(\mathcal{P} \otimes M) \otimes (\mathcal{P}' \otimes M') \longrightarrow (\mathcal{P} \otimes \mathcal{P}') \otimes (M \otimes M')$$

(here the tensor product on the right is defined over $G \times G'$) and

$$\mathrm{Hom}(\mathcal{P}, M) \otimes \mathrm{Hom}(\mathcal{P}', M') \longrightarrow \mathrm{Hom}(\mathcal{P} \otimes \mathcal{P}', M \otimes M')$$

where the Hom functor on the right is taken over $G \times G'$ (see Chapter 5 in [4] and/or Chapter 6 in [16]). From these maps we obtain

$$H_p(G, M) \otimes H_q(G', M') \to H_{p+q}(G \times G', M \otimes M'),$$
$$H^p(G, M) \otimes H^q(G', M') \to H^{p+q}(G \times G', M \otimes M')$$

called the homology and cohomology ***cross-product***, respectively. These products are called ***external*** products since they involve the three groups $G, G', G \times G'$. In what follows we concentrate on the case $G = G'$, in such case the products above are called ***internal***.

Consider the diagonal map $d : G \to G \times G$ and compose the cohomology cross-product above with the induced homomorphism on cohomology d^* to get the ***cup product***:

$$H^p(G, M) \otimes H^q(G, M') \to H^{p+q}(G, M \otimes M'), \quad u \otimes v \longmapsto d^*(u \times v)$$

The cup product of u and v is denoted by $u \cup v$ or simply uv. In what follows we will list some of the properties of the cup product

(1) By the properties mentioned on the last section in dimension 0 the cup product

$$H^0(G; M) \otimes H^0(G; N) \longrightarrow H^0(G; M \otimes N)$$

is given by the map $M^G \otimes N^G \to (M \otimes N)^G$, that it is induced by the inclusions $M^G \to M, \quad N^G \to N$.

(2) Let $f : M \to M', g : N \to N'$ be homomorphisms of G-modules. For elements $u \in H^*(G; M), v \in H^*(G; N)$ one has

$$(f \otimes g)_*(u \cup v) = f_* u \cup g_* v \in H^*(G; M' \otimes N'),$$

where f_* stands for the induced homomorphism on the coefficients module.

(3) Let $0 \to M' \to M \to M'' \to 0$ a short exact sequence of G-modules and let N be a module such that $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact. Then the cup product is compatible with the connecting homomorphism δ in the sense that the following diagram commutes

$$\begin{array}{ccc} H^p(G; M'') & \xrightarrow{\delta} & H^{p+1}(G; M') \\ \downarrow & & \downarrow \\ H^{p+q}(G; M'' \otimes N) & \xrightarrow{\delta} & H^{p+q+1}(G; M' \otimes N) \end{array}$$

where vertical maps are $- \cup v$ the cup product with v. In other words, one has $\delta(u \cup v) = \delta u \cup v$, for $u \in H^p(G; M'')$ and $v \in H^q(G; N)$. See Section V.3 of [4].

(4) The element $1 \in H^0(G,\mathbb{Z}) = \mathbb{Z}$ satisfies
$$1 \cup u = u = u \cup 1$$
for all $u \in H^*(G,M)$. (Implicity one needs the identification $\mathbb{Z} \otimes M = M = M \otimes \mathbb{Z}$.)

(5) Consider $u_i \in H^*(G,M_i)$, for $i = 1,2,3$. Then one has
$$(u_1 \cup u_2) \cup u_3 = u_1 \cup (u_2 \cup u_3) \in H^*(G, M_1 \otimes M_2 \otimes M_3)$$

(6) For elements $u \in H^p(G,M), v \in H^q(G,N)$ one has
$$u \cup v = (-1)^{pq} t_*(v \cup u) \in H^{p+q}(G, M \otimes N)$$
where t_* is the induced homomorphism by $t : N \otimes M \to M \otimes N$, $n \otimes m \mapsto m \otimes n$.

All the properties above prove the following

Proposition 4.2. *$H^*(G;\mathbb{Z})$ is an anti-commutative graded ring and $H^*(G,M)$ is a graded module over $H^*(G;\mathbb{Z})$, for every G-module M.*

Moreover, if k denotes a commutative ring (with the trivial G-action) then the cup product makes $H^*(G,k)$ a graded anti-commutative k-algebra.

4.2. **More on free actions on spheres.** Recall from Section 2.4 that if a finite group G acts freely on a odd-dimensional sphere S^{2k-1} then there is a periodic resolution of $\mathbb{Z}$ of period $2k$. Considering this resolution we find that there is an iterated coboundary map
$$d : H^i(G,M) \longrightarrow H^{i+2k}(G,M)$$
which is an isomorphism for $i > 1$ and an epimorphism for $i = 1$, for any G-module M. Moreover, by the properties of the cup product $d(w \cup v) = d(w) \cup v$, for any $w \in H^*(G,\mathbb{Z})$ and $v \in H^*(G,M)$. For $w = 1$ this implies that
$$d(v) = u \cup v$$
for a certain class $u \in H^{2k}(G,\mathbb{Z})$. This will be called the ***periodicity isomorphism*** when we define periodic cohomology in next section.

5. Groups with periodic cohomology

Tate's cohomology theory is a way to obtain a cohomology theory that packs both homology and cohomology of a finite group into a single functor $\widehat{H}^*(G)$ that it is constructed by gluing the groups $H_0(G)$ and $H^0(G)$ via ***complete resolutions*** which are resolutions with indices in $\mathbb{Z}$ rather than in $\mathbb{N}$. The following shows this procedure

$$\begin{array}{cccccccc} \cdots & H_2 & H_1 & \widetilde{H}_0 & \widetilde{H}^0 & H^1 & H^2 & \cdots \\ & \| & \| & \| & \| & \| & \| & \\ \cdots & \widehat{H}^{-3} & \widehat{H}^{-2} & \widehat{H}^{-1} & \widehat{H}^0 & \widehat{H}^1 & \widehat{H}^2 & \cdots \end{array}$$

where $\widetilde{H}_0 \subset H_0$ and $\widetilde{H}^0$ is certain quotient of H^0. The theory $\widehat{H}^*$ can also be described axiomatically but we will not give that description here (see [4], Chapter 6). For the purposes of the present notes it is enough to know that there is a cohomological group theory that "generalizes" both homology and cohomology of a group in the sense of the isomorphisms

$$\begin{cases} \widehat{H}^k(G) \cong H^k(G), & k = 1, 2, 3, \ldots \\ \widehat{H}^k(G) \cong H_{-(k+1)}(G), & k = -1, -2, -3, \ldots \end{cases}$$

Also, the importance of Tate cohomology is the existence of invertible cohomological classes of positive degree, which is something we cannot have in ordinary group cohomology. The existence of invertible elements gives rise to the phenomenon of cohomological periodicity.

A finite group G is said to be ***periodic*** or to have ***periodic cohomology*** if for some $d \neq 0$ there exists $u \in \widehat{H}^d(G, \mathbb{Z})$ which is invertible in the ring $\widehat{H}^*(G, \mathbb{Z})$. The existence of this element gives rise to the ***periodicity isomorphism***

$$\widehat{H}^n(G, \mathbb{Z}) \xrightarrow{\cong} \widehat{H}^{n+d}(G, \mathbb{Z})$$

via the cup product. The number d is called the ***period*** of G.

It is obvious that for periodic groups the task of computing their cohomology is enormously simplified, thus one looks for conditions on a group for having periodic cohomology. One condition is immediate: if a group admits a periodic resolution then it has periodic cohomology. This is the case for the finite cylic group $\mathbb{Z}/n\mathbb{Z}$ via the resolution obtained in Section 2.4. The property of having periodic cohomology can be stated in different ways as the following proposition shows

Proposition 5.1 ([4])**.** *The following conditions are equivalent*

(1) *G has periodic cohomology*
(2) *There exists integers $n, d \neq 0$ such that $\widehat{H}^n(G, M) \cong \widehat{H}^{n+d}(G, M)$, for all G-modules M.*
(3) *For some $d \neq 0$, $\widehat{H}^d(G, \mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$*
(4) *For some $d \neq 0$, $\widehat{H}^d(G, \mathbb{Z})$ contains an element u or order $|G|$.*

5.1. **Periodicity via actions on spheres.** Recall in Section 2.4 it was shown when a group G acts on S^1 then it has a periodic resolution and thus periodic cohomology. This result can be generalized as follows

Proposition 5.2. *Let G be acting freely and orientation-preserving on a sphere S^{n-1}. Then G has periodic cohomology of period n.*

The proof of this result follows as in Section 2.4: the G-action produces a periodic resolution from which one obtains periodic cohomology. An alternative proof can be obtained as follows: from the G-action one has a fibration

$$S^{n-1} \longrightarrow S^{n-1}/G \longrightarrow BG$$

whose spectral sequence in cohomology converges to $H^*(S^{n-1}/G)$. Since G acts orientation-preserving on S^{n-1} (considered as a free G-complex) we can check that G acts trivially on $H_*(S^{n-1}), H^*(S^{n-1})$ (by applying the Lefschetz Fixed-Point Theorem). Thus, in the E_2-term

$$E_2^{i,j} = H^i(BG, H^j(S^{n-1})) \cong H^i(G, H^j(S^{n-1}))$$

one has a trivial G-action on the module of coefficients (in this case one also says that the ***system of local coefficients*** is trivial). The E_2-term of the spectral sequence has only two non-trivial rows and thus a unique differential

$$d_n : E_2^{i,n-1} \cong H^i(G, \mathbb{Z}) \longrightarrow E_2^{i+n,0} \cong H^{i+n}(G, \mathbb{Z})$$

One finally proves d_n is the periodicity isomorphism of G by considering that the target $H^*(S^{n-1}/G)$ has trivial cohomology for $* > n-1$, since the quotient S^{n-1}/G is $(n-1)$-dimensional. See [1] for more details.

In what follows we will review the construction of quaternionic numbers in order to talk about the topological group S^3 and some of its subgroups.

5.1.1. *Subgroups of S^3.* Let $\mathbb{C}$ denote the set of complex numbers. The ***quaternions*** $\mathbb{H}$ are formed as the pairs $\mathbb{C} \oplus \mathbb{C}$ of complex numbers with multiplication given by the formula

$$(z_1 + z_2 j) \cdot (z_3 + z_4 j) = (z_1 z_3 - z_2 \overline{z}_4) + (z_1 z_4 + z_2 \overline{z}_3) j,$$

where $\overline{z}$ is the conjugation of z and j is an imaginary unit ($j^2 = -1$). This multiplication is associative and has as identity the element $1 = 1 + 0j$. For a quaternion $\alpha = z_1 + z_2 j$ one define the ***conjugation map*** as

$$\omega : \mathbb{H} \longrightarrow \mathbb{H}, \qquad \omega(\alpha) = \overline{z}_1 - z_2 j,$$

This map has the following properties: for any $\alpha \in \mathbb{H}$

- $\alpha\omega(\alpha) = \omega(\alpha)\alpha$
- $\alpha\omega(\alpha)$ is contained in the center of $\mathbb{H}$, which is $\mathbb{R}$.
- $\omega(\alpha\beta) = \omega(\alpha)\omega(\beta)$.

The ***norm map*** is defined as

$$N : \mathbb{H} \longrightarrow \mathbb{R}^{>0} \qquad N(\alpha) = \omega(\alpha)\alpha$$

It is easy to note that this defines a homomorphism $N : \mathbb{H} \backslash \{0\} \to \mathbb{R}$ and the subgroup $N^{-1}(1)$ of unit quaternions is identified with S^3, the 3-sphere. Moreover, one can construct a map $\varphi : S^3 \to SO(3)$ with kernel $\mathbb{Z}_2$ (See [1], Section 1.2). That is, one has an extension

$$\mathbb{Z}_2 \longrightarrow S^3 \xrightarrow{\varphi} SO(3)$$

More extensions can be constructed from this by applying the following

Lemma 5.3. *Let $N \to G \xrightarrow{\varphi} G/N$ be a given extension with N finite. For a subgroup $V \subset G/N$ there is an extension $N \to \varphi^{-1}(V) \to V$. If V is finite so is $\varphi^{-1}(V) \subset G$ and $|\varphi^{-1}(V)| = |N||V|$.*

From this lemma one gets the following family of subgroups of S^3:

(1) Let

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

with $\theta = 2\pi/n$, for some integer n. The subgroup generated by α, β is the dihedral group D_{2n} of order $2n$ (This group can also be considered as the group of symmetries of a n-gon). The **generalized quaternion** group is the pre-image $Q_{4n} = \varphi^{-1}(D_{2n})$ and one has

$$\mathbb{Z}_2 \to Q_{4n} \to D_{2n}$$

For $n = 2$, one has the quaternion extension $\mathbb{Z}_2 \to Q_8 \to \mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) The **tetrahedral** group T is the group of rotations of a regular tetrahedron; it has order 12 and is isomorphic to A_4. The **binary tetrahedral** group is $T^* = \varphi^{-1}(T)$ and thus one has

$$\mathbb{Z}_2 \longrightarrow T^* \longrightarrow T$$

(3) The **octahedral** group O is the group of rotations of a regular octahedron; it has order 24 and is isomorphic to Σ_4. The **binary octahedral** group is $O^* = \varphi^{-1}(O)$ and thus there is an extension:

$$\mathbb{Z}_2 \longrightarrow O^* \longrightarrow O$$

(4) The **icosahedral** group I is the group of rotations of a regular icosahedron; it has order 60 and is isomorphic to A_5. The **binary icosahedral** group is $I^* = \varphi^{-1}(I)$ and thus there is an extension:

$$\mathbb{Z}_2 \longrightarrow I^* \longrightarrow I$$

Since S^3 is a continuous group, all these subgroups act freely on S^3 and thus all have periodic cohomology of period 4.

Lemma 5.4. *Generalized quaternions, binary tetrahedral, binary octahedral and binary icosahedral all have periodic cohomology.*

A different approach for this result is given in Example 2, p. 155 of [4] where all these groups are shown to have periodic cohomology of period 4 by means of a free action on certain even-dimensional vector space.

The converse of the Proposition 5.2 above is not true: there are groups with periodic cohomology that do not act freely on any sphere. One example is the symmetric group Σ_3 on 3 letters whose cohomology is given next (see [1], p. 147 for details)

$$H^*(\Sigma_3, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/2, & i = 2(\text{mod } 4) \\ \mathbb{Z}/6, & i = 0(\text{mod } 4) \\ 0, & i \text{ odd} \end{cases}$$

One can also prove that this group does not have periodic cohomology by showing it cannot act freely on any sphere. This can be obtained by the following generalization of the Borsuk-Ulam theorem due to J. Milnor

Theorem 5.5 ([12]). *Let $n > 0$ and let $T : S^n \to S^n$ be a continuous map without fixed points such that $T \circ T = 1_{S^n}$. Then, for every continuous map $f : S^n \to S^n$ of odd degree there exists $x \in S^n$ with*

$$T \circ f(x) = f \circ T(x)$$

(Note when the map T is the antipodal map the theorem above is simply the Borsuk-Ulam).

From this result it follows that any finite group acting freely on a sphere can contain at most one element of order 2 ([11]) and thus the symmetric group Σ_n cannot act freely on a sphere if $n \geq 3$ since every transposition is an element of order 2.

5.2. **Periodicity detected by subgroups.** In this final section we will see that the property of having periodic cohomology can be stated in terms of the presence of certain subgroups. First, it is worth noting that if a group G has periodic cohomology then any of its subgroups also has periodic cohomology since via the inclusion $H \subset G$ the induced map $H^*(G, \mathbb{Z}) \to H^*(H, \mathbb{Z})$ is a ring homomorphism that takes invertible elements into invertible elements.

The next example shows that in general the property of having periodic cohomology is not invariant under products

Example 5.6. *Let p any prime and recall from the introduction that if p is odd, then $H^*(\mathbb{Z}_p) = \Lambda(x) \otimes \mathbb{F}_p[y]$ and for $p = 2$ one has $H^*(\mathbb{Z}_2) = \mathbb{F}_2[x]$. Also recall that the classifying space for a product of groups $G \times H$ is the product of the associated classifying spaces $BG \times BH$, thus one has*

$$H^*((\mathbb{Z}_p)^n; \mathbb{F}_p) = \begin{cases} \mathbb{F}_p[x_1, x_2, \ldots, x_n], & p = 2 \\ \Lambda(x_1, x_2, \ldots, x_n) \otimes \mathbb{F}_p[y_1, y_2, \ldots, y_n], & p \ \ odd \end{cases}$$

where $|x_i| = 1, |y_j| = 2$, for all i, j. In particular, this shows that the cohomology of the product $(\mathbb{Z}_p)^n$ is not periodic. One can also consider *the homology of* $\mathbb{Z}_n \times \mathbb{Z}_n$, *for* $n \in \mathbb{N}$, *via the Kunneth formula*

$$H_k(G \times G', M) \cong \bigoplus_{p+q=k} H_p(G,M) \otimes H_q(G',M) \bigotimes \bigoplus_{p+q=k-1} Tor_1(H_p(G,M), H_q(G',M))$$

to obtain

$$H_k(\mathbb{Z}_n \times \mathbb{Z}_n; \mathbb{Z}) \cong \bigoplus_{k/2+1} \mathbb{Z}_n$$

which shows the homology of $\mathbb{Z}_n \times \mathbb{Z}_n$ *is not periodic. Finally, with the identification of* $\widehat{H}^k(G) \cong H_{-(k+1)}(G)$ *one has that the product* $\mathbb{Z}_n \times \mathbb{Z}_n$ *does not have periodic Tate cohomology. See* [11], *Section 1.9.*

We will finish the present notes by noting that the presence of certain subgroups determines the periodicity of a group as the following theorem shows.

Theorem 5.7 (Theorem 9.5 in [4])**.** *The following conditions are equivalent for a finite group G:*

1. *G has periodic cohomology*
2. *Every abelian subgroup of G is cyclic*
3. *Every elementary abelian p-subgroup of G has rank* ≤ 1.
4. *The Sylow subgroups of G are cyclic or generalized quaternion groups.*

In particular, this theorems says that if a certain group G contains (a copy of) $\mathbb{Z}_p \times \mathbb{Z}_p$, for a prime p, then G does not have periodic cohomology. Thus, in practical situations, when one tries to prove that a certain group G has periodic cohomology one looks at its abelian subgroups.

Miguel A. Maldonado
Laboratoire de Mathématiques Nicolas Oresme,
Université de Caen Basse-Normandie.

References

[1] Adem, A., Milgram, R.J. , **Cohomology of Finite Groups**, Springer-Verlag, 1994.

[2] Arkowitz, M., **An Introduction to Homotopy Theory**, Springer-Verlag, 2011.

[3] Benson, **Representations and Cohomology II**, Cambridge Studies in Advanced Mathematics 31, 2004.

[4] Brown, **Cohomology of Groups**, Springer-Verlag, 1982.

[5] Dummit, D.S., Foote, R.M., **Abstract Algebra**, 3rd edition, John Wiley and Sons Inc., 2004.

[6] Hatcher, A., **Algebraic Topology**, Cambridge University Press, 2002.

[7] Hilton, P.J., Stammbach, U., **A Course in Homological Algebra**, Graduate Texts in Mathematics **4**, Springer-Verlag.

[8] Hopf, H., *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. **14** (1942), 257-309.

[9] Hurewicz, W., *Beiträge zur Topologie der Deformationen. IV. Asphärische Räume*, Nederl. Akad. Wetensch. Proc. **39** (1936), 215-224.

[10] Husemoller, **Fiber Bundles**, Springer-Verlag,.

[11] Löh, C., Group Cohomology and Bounded Cohomology. An Introduction for Topologists. `http://www.math.uni-muenster.de/u/clara.loeh/`

[12] Milnor, J., *Groups which act on S^n without fixed points*, Amer. J. Math., **79** (1957), 623-630.

[13] Milnor, J., *Construction of Universal Bundles II*, Annals of Mathematics. **63** 3 (1956), 430-436.

[14] Rotman, J.J., **An Introduction to Homological Algebra**, Springer, Universitext .

[15] Webb, P.J., **An introduction to the cohomology of groups**.

[16] Weibel, C.A., **An Introduction to Homological Algebra**,